CO-ORDINATE GEOMETRY

CARTESIAN SYSTEM OF RECTANGULAR COORDINATES

Any point in the plane have Cartesian coordinates (x, y), where x is called **abscissa** and y is called **ordinate.**

(A) Co-ordinates of the origin are (0, 0).

(B) Any point on the x-axis has ordinate = 0 i.e. is in the form (x, 0)

(C) Any point on the y-axis has abscissa = 0 i.e. is in the form (0, y)

SOME RESULTS:

(i) DISTANCE FORMULA

If A (x_1, y_1) and B (x_2, y_2) be two points, then

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In particular, distance of a point P (x, y) from O (0, 0) is:

$$|OP| = \sqrt{x^2 + y^2}$$

(ii) SECTION FORMULA

The point which divides the join of two distinct points A (x_1, y_1) and B (x_2, y_2) in the ratio $m_1 : m_2$

- **Internally**, is

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right), \quad m_1 \neq 0,\ m_2 \neq 0,\ m_1 + m_2 \neq 0$$

- **Externally**, is

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right) \quad m_1 \neq 0,\ m_2 \neq 0,\ m_1 - m_2 \neq 0$$

In particular, the mid-point of the segment joining A $(x_1\ y_1)$ and B (x_2, y_2) has the co-ordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

(iii) CENTROID AND IN-CENTRE FORMULAE

If A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) be the vertices of a triangle

Centroid: It is the point of intersection of the medians of a triangle, & is given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

In-centre: It is the point of intersection of the internal angle bisectors of the angles of a triangle, & is given by

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

Where a = | BC |, b = | CA | and c = | AB |.

(iv) AREA OF TRIANGLE

If A (x_1, y_1), B (x_2, y_2) and C(x_3, y_3) are the vertices of a triangle then,

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\}$$

Note:- Three points A, B & C are **collinear**, if area of the triangle ABC is zero.

(v) AREA OF QUADRILATERAL

If A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) and D (x_4, y_4) be the vertices of a quadrilateral, then,

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}\ |\ (x_1 y_2 - x_2y_1) + (x_2 y_3 - x_3 y_2) + (x_3y_4 - x_4y_3) + (x_4 y_1 - x_1y_4)\ |$$

(iv) TRANSLATION OF AXES

A change of origin without changing the direction of coordinate axes is called a translation of axes.

If the coordinates of a point change from (x, y) to (X, Y) when the origin is shifted to the point (h, k), axis remaining parallel to the original axis, then

$X = x - h$, $Y = y - k$

Or $x = X + h$, $y = Y + k$

Remark: The above transformation equations also hold good even if h or k or both h and k are not positive.

STRAIGHT LINE

(i) **Equation of any line parallel to X-axis** is y = b,
b being the directed distance of the line from the X-axis
In particular equation of X-axis is y = 0

(ii) **Equation of any line parallel to Y-axis** is x = a,
a being the directed distance of the line from the Y-axis.
In particular equation of Y-axis is x = 0.

(iii) **Slope** of a non-vertical line L is the tangent of the angle θ, which either of half ray of the line L makes with the positive direction of X-axis.

- Slope of the line $ax + by + c = 0$, $b \neq 0$, is $-\frac{a}{b}$
- Slope of the line through the points A (x_1, y_1) and B (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$

In particular,

(a) Slope of a line parallel of X-axis is 0.
(b) Slope of a line parallel to Y-axis is not defined.
(c) Slope of a line equally inclined to both the axis is –1 or 1.
(d) Slope of a line making equal intercepts on both the axis is –1.

NOTE:

- If slopes of two parallel (non-vertical) lines are equal then $m_1 = m_2$
- If m_1 and m_2 be the slopes of two perpendicular lines (which are oblique), then $m_1 m_2 = -1$.
- Equation of line (non- vertical) through the origin and having slope m is $\mathbf{y = mx}$.

DIFFERENT FORMS OF EQUATION OF A LINE

☞ POINT SLOPE FORM

Equation of a line (non-vertical) through the point (x_1, y_1) and having slope m is

$$y - y_1 = m(x - x_1)$$

Equation of the vertical line through the point (x_1, y_1) is $\mathbf{x = x_1}$

☞ SLOPE –INTERCEPT FORM

Equation of a line (non-vertical) with slope m and cutting off an intercept c from the Y-axis is

$$y = mx + c$$

☞ INTERCEPT FORM

Equation of a line (non-vertical) with slope m and cutting off intercepts a and b from the x-axis and y-axis respectively is

$$\frac{x}{a} + \frac{y}{b} = 1$$

☞ TWO POINT FORM

Equation of the line through the points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1), \quad x_1 \neq x_2$$

In particular if $x_1 = x_2$, then the line is vertical and its equation is $\mathbf{x = x_1}$ **or** $\mathbf{x = x_2}$.

☞ DISTANCE FORM

Equation of line through the point (x_1, y_1) & making an angle θ with the positive directions of X-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

Where $|r|$ being the distance of (x, y) from (x_1, y_1).

☞ NORMAL or PERPENDICULAR FORM

Equation of a line not passing through the origin is

$$x\cos\alpha + y\sin\alpha = p$$

Where p is the length of perpendicular from the origin upon the line and α is the angle which this perpendicular (directed away from the origin) makes with the positive direction of X-axis.

CONCURRENCY OF THREE STRAIGHT LINES

The three straight lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

SOME IMPORTANT RESULTS

- **Length of perpendicular** from the point (x_1, y_1) to the line $ax + by + c = 0$ is

$$L = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

- **Distance between || lines ax + by + c = 0 and ax + by +d =0**

$$\frac{|c - d|}{\sqrt{a^2 + b^2}}$$

- **The angle between two lines** $y = m_1x + b_1$ and $y = m_2x + b_2$ is given by

$$\tan\theta = \pm\left(\frac{m_2 - m_1}{1 + m_1m_2}\right),\ m_1m_2 \neq -1$$

In particular,

The **acute angle** θ is given by $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1m_2}\right|$

The **obtuse angle** θ is given by $\tan\theta = -\left|\frac{m_2 - m_1}{1 + m_1m_2}\right|$

In other form,

An angle θ between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\tan\theta = \pm\left(\frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}\right), a_1a_2 + b_1b_2 \neq 0.$$

In particular,

The **acute angle** θ given by $\tan\theta = \left|\frac{a_1b_2 - a_2b_1}{a_1b_2 - b_1b_3}\right|$

The **obtuse angle** θ is given by $\tan\theta = -\left|\frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}\right|$

- **The equation of a straight line passing through (x_1, y_1) and making an angle α with the line y = mx + c are**

$$y - y_1 = \frac{m - \tan\alpha}{1 + m\tan\alpha}(x - x_1) \text{ and } y - y_1 = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - x_1)$$

NOTE: -

☞ The equation $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ represent the same line if $a_2 = \lambda a_1$, $b_2 = \lambda b_1$ and $c_2 = \lambda c_1$ for some none zero real number λ.

☞ The equation of any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, i.e., there is change only in constant term.

☞ The equation of any line perpendicular to $ax + by + c = 0$ is of the type $bx - ay + k = 0$ i.e., interchange the coefficient of x and y and change the constant term .

☞ Two lines $a_1 + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ intersect iff $a_1b_2 - a_2b_1 \neq 0$.

The point in which they intersect is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

☞ A line through the point of intersection of $l_1 = 0$ and $l_2 = 0$ is usually taken as $l_1 + kl_2 = 0$,

☞ The equation $y - y_1 = m (x - x_1)$, $m \in R$, represents the family of all lines through the point (x_1, y_1) except the vertical line.

Orthocentre & Circumcentre :

☞ The Orthocentre is the point of intersection of the altitudes from vertices to the opposite sides of triangles and the Circumcentre is the point of intersection of perpendicular bisectors of the sides of triangle.

EXAMPLES

Ex. Find the coordinates of the point which divides

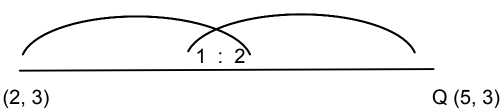
(i) The join of (2, 3) and (5, – 3) internally in the ratio of 1 : 2.

(ii) The join of (2, 1) and (3, 5) externally in the ratio 2 : 3.

Sol. (i) Let the given points be P (2, 3) and Q (5, – 3). Let R (x, y) be the required point.

$$\therefore \quad x = \frac{1\times 5 + 2\times 2}{1+2} = \frac{9}{3} = 3$$

$$y = \frac{1\times(-3) + 2\times 3}{1+2} = \frac{3}{3} = 1$$

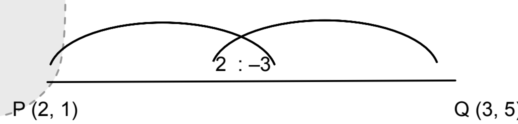


$\therefore$ R (3, 1) divides PQ internally in the ratio 1 : 2.

(ii) Let the given points be P (2, 1) and Q (3, 5). Let R (x, y) be the required point. To find x, y we use the formula for internal division for the ratio 2 : – 3.

$$\therefore \quad x = \frac{2\times 3 + (-3)\times 2}{2+(-3)} = \frac{6-6}{-1} = 0$$

$$y = \frac{2\times 5 + (-3)\times 1}{2+(-3)} = \frac{10-3}{-1} = -7$$



$\therefore$ The required points is (0, – 7).

Ex. In what ratio is the line joining (4, 5) and (1, 2) is divided by the y–axis. Also find the point of division, and show it on a coordinate plane.

Sol. Let the given pints be P (4, 5) and Q (1, 2).

Let PQ be divided by the y–axis at the point R (0, Y) in the ratio k : 1 internally.

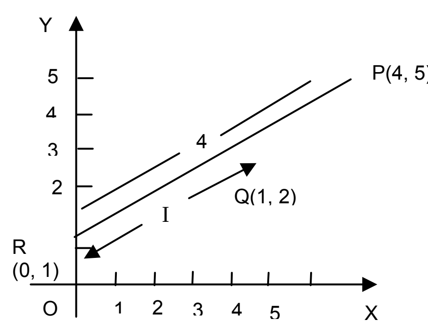
$$0 = \frac{1x4 + kx1}{k+1} \qquad \text{(Considering the abscissa of R)}$$

$\Rightarrow k = -4.$

$\therefore$ The ratio is – 4 : 1 internal, i.e., 4 : 1 External.

$$\therefore y = \frac{1\times 5 + k\times 2}{k+1} = \frac{5 - 4\times 2}{-4+1} = 1.$$

$\therefore$ The required point is (0, 1)



Ex. Find the gradient of the straight line joining the points (2, 3) and (– 6, 7).

Sol. m = (3 – 7)/(2 + 6) = – 1/2

Ex. Find the equation of the line with slope 2 and X-intercept – 3.

Sol. The line is 2(X + 3) = Y – 0 $\Rightarrow$ 2X + 6 = Y $\Rightarrow$ 2X – Y + 6 = 0

Ex. Find the equation of the straight line passing through the point (2, 1) and parallel to the line through the points (– 3, – 2) and (6, 5).

Sol. The slope of the line through (3, – 2) and (– 6, 5) is m = (5 + 2) / (– 6 – 3) = – 7/9.
By point – slope form, the equation of the line through (2, 1) with slope – 7/9 is
Y – 1 = – 7/9 (X – 2) $\Rightarrow$ 9Y – 9 = – 7X + 14 $\Rightarrow$ 7X + 9Y – 23 = 0

Ex. A (– 3, 3), B (5, 5), C (2, –4) are vertices of triangles ABC, find the equation of the altitude from A to BC.

Sol. Slope of BC = (5 + 4) / (5 – 2) = 3.
$\Rightarrow$ Slope of the altitude = – 1/3.
$\Rightarrow$ By point-slope form – 1/3 (X + 3) = Y – 3 $\Rightarrow$ X + 3Y – 6 = 0

Ex. Find the equation of the line which passes through the point of intersection of the lines.
7X – 9Y + 31 = 0 and 11X + 3Y – 37 = 0 and is perpendicular to the line 3X + Y – 2 = 0.

Sol. Let the required line be (7X – 9Y + 31) + K (11X + 3Y – 37) = 0
$\Rightarrow$ (7 + 11K) X + (3K – 9) Y + 31 – 37 K = 0
$\Rightarrow$ The slope of the line which passes through the point of intersection of the lines 7X – 9Y + 31 = 0 and 11X + 3Y – 37 = 0 is $-\frac{7+11k}{9-3k}$ and the slope of the line 3X + Y – 2 = 0 is –3
$\Rightarrow -\frac{7+11k}{9-3k}$ (–3) = 1 or (7 + 11k)/(9 – 3k) = 1/3
$\Rightarrow$ The required equation is (7X – 9Y + 31) – 1/3 (11X + 3Y – 37) = 0
$\Rightarrow$ X – 3Y + 13 = 0

Ex. The equation of a straight line passing through (3, 4) and having sum of intercepts as 14 is:
(A) x + y = 7, 4x + 3y = 24 (B) x + y = 8, 4x + 3y = 21
(C) x + 2y = 11, 2x – y – 2 = 0 (D) None of these

Sol. Let intercepts on x-axis and y-axis are a and b respectively.
$\therefore$ The equation of the line is $\frac{x}{a}+\frac{y}{b}=1$ … (1)
As (3, 4) lie on it
$\therefore \frac{3}{a}+\frac{4}{b}=1$ or, 3b + 4a = ab ... (2)
Also given that a + b = 14 … (3)
Substituting the value of b in (2) from (3), we get 3(14 – a) + 4a = a (14 – a)
$\Rightarrow$ (a – 6) (a – 7) = 0 $\Rightarrow$ a = 6, 7
When a = 6, b = 8, the equation of line $\frac{x}{6}+\frac{y}{8}=1$ or 4x + 3y = 24
When a = 7, b = 7, the equation of line is $\frac{x}{7}+\frac{y}{7}=1$ or x + y = 7 The correct answer is (A).

Ex. If the centroid of a triangle with vertices (x, – 6), (2, y) and (2, 5) is (8/3, – 1), then the values of x and y are respectively

(A) 2 and 4 (B) – 2 and 4 (C) 2 and – 4 (D) 4 and –2

Sol. The centroid of the triangle with vertices A (x, – 6), B (2, y) and C (2, 5) is given by

$\frac{8}{3} = \frac{x+2+2}{3} \Rightarrow x = 4$ and $-1 = \frac{-6+y+5}{3} \Rightarrow y = -2$

∴ The correct answer is (D).

Ex. The line $x - \sqrt{3}y + 2 = 0$ expressed in the normal form is:

(A) $x \cos 60^o + y \sin 60^0 = 1$ (B) $x \cos 120^o + y \sin 120^o = 1$

(C) $x \cos 150^o + y \sin 150^o = 1$ (D) none of these

Sol. The given equation of the straight line is $x - \sqrt{3}y + 2 = 0$

or $-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 1$

or $x \cos 120^o + y \sin 120^o = 1$.

The correct answer is (B).

Ex. The distance between the straight lines 6x + 8y = 15 and 3x + 4y = 9 is :

(A) 3/2 (B) ¾ (C) 3/10 (D) 2/7

Sol. Perpendicular length from origin to the line 3x + 4y = 9 is: $p_1 = \frac{3.0+4.0-9}{\sqrt{3^2+4^2}} = \frac{-9}{5}$

Perpendicular length from origin to the line 6x + 8y = 15 is: $p_2 = \frac{6.0+8.0-15}{\sqrt{6^2+8^2}} = \frac{-15}{10} = -\frac{3}{2}$

⇒ If p_1 and p_2 are of same sign then distance between two lines = $p_1 - p_2$

⇒ If p_1 and p_2 are of opposite sign then the distance between two lines = $p_1 + p_2$

Hence in this question p_1 and p_2 are of same sign.

∴ Distance between the given lines = $p_2 - p_1 = -3/2 - (-9/5) = 3/10$.

∴ The correct answer is (C).

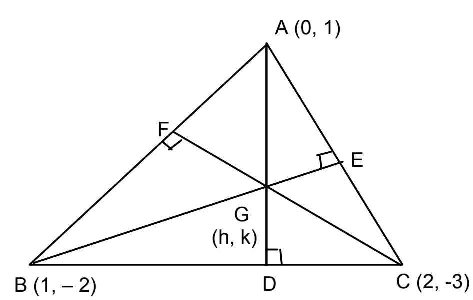
Ex. Find the coordinate of the orthocenter of the triangle whose vertices are (0, 1), (1, – 2), (2, – 3).

Sol. Let the vertices of the triangle be A (0, 1), B (1, – 2) and C (2, – 3).

The orthocentre of the triangle is the point of concurrence of the altitudes from the vertices.

Let AD, BE and CF be the altitudes, where G (h, k) is the orthocentre of the triangle.

AG ⊥ BC ⇒ slope of AG × slope of BC = – 1.



$\Rightarrow \quad \frac{k-1}{h-0} \times \frac{-3+2}{2-1} = -1$

$\Rightarrow \quad -k + 1 = -h \Rightarrow h - k + 1 = 0.$ … (1)

Also, BG ⊥ AC ⇒ $\frac{k+2}{h-1} \times \frac{-3-1}{2-0} = -1$

$\Rightarrow -4k - 8 = -2h + 2 \qquad \Rightarrow h - 2k - 5 = 0$ … (2)

$(1) - (2) \Rightarrow k + 6 = 0 \qquad \Rightarrow k = -6$

$\therefore (1) \Rightarrow h - (-6) + 1 = 0 \qquad \Rightarrow h = -6$

$\therefore (1) \Rightarrow h - (-6) + 1 = 0 \qquad \Rightarrow h = -7$

∴ The orthocentre is (– 7, – 6).

Ex. The equation of lines passing through (3, – 2) and making an angle of 60^0 with the line $\sqrt{3}x + y = 1$ are

(A) $y = -2,\ y = \sqrt{3}x + 2 + \sqrt{3}$ (B) $y = -2,\ 3y = \sqrt{3}x - 6 - 3\sqrt{3}$

(C) $y + 2 = 0,\ y = \sqrt{3}x - 2 - 3\sqrt{3}$ (D) None of these

Sol. Here $\alpha = 60^0$, $m = -\sqrt{3}$ and point (3, – 2).

∴ The equation of lines are given by

$$y - (-2) = \frac{(-\sqrt{3}) \pm \tan 60^0}{1 \pm (-\sqrt{3})\tan 60^0}(x - 3) \Rightarrow y + 2 = \frac{-\sqrt{3} \pm \sqrt{3}}{1 \pm (-\sqrt{3})(\sqrt{3})}(x - 3)$$

⇒ Taking negative sign in numerator, we get

$$y + 2 = \frac{-2\sqrt{3}}{-2}(x - 3) \Rightarrow y + 2 = \sqrt{3}x - 3\sqrt{3} \text{ or } y = \sqrt{3}x - 2 - 3\sqrt{3}.$$

⇒ Taking positive sign in numerator, we get

$y + 2 = 0/4(x - 3)$ or $y + 2 = 0$

∴ The correct answer is (C)

Ex. If the straight lines $x + y - 4 = 0$, $3x + 2 = 0$ and $x - y + 3k = 0$ are concurrent, then the value of k is:

(A) 8/9 (B) – 12/9 (C) 12/9 (D) 16/9

Sol. The straight lines $x + y - 4 = 0$, $3x + 2 = 0$ and $x - y + 3k = 0$ will be concurrent if:

$$\begin{vmatrix} 1 & 1 & -4 \\ 3 & 0 & 2 \\ 1 & -1 & 3k \end{vmatrix} = 0 \Rightarrow 1(0 + 2) - 1(9k - 2) - 1(9k - 2) - 4(-3 - 0) = 0 \Rightarrow 9k - 16 = 0 \Rightarrow k = 16/9.$$

The correct answer is (D).

BISECTOR OF ANGLES

The equations of straight lines bisecting the angles between the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are

$$\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right) \quad \text{…. (1)}$$

If c_1 and c_2 are of same sign then,

- **Positive sign** shows the bisector of angle in which origin lie.
- **Negative sign** shows the bisector of angle in which origin does not lie.

Note: if $a_1 a_2 + b_1 b_2 < 0$ + ve sign for acute angle
– ve for obtuse

if $a_1 a_2 + b_1 b_2 > 0$ +ve sign for obtuse angle,
–-ve for acute angle.)

In which origin lies take + ve sign

Example

Ex. The equation of angle of bisector of the equations $3x + 4y + 7 = 0$ and $12x + 5y - 4 = 0$ in which origin does not lie is:

(A) $99x + 77y + 71 = 0$ (B) $7x - 9y = 37$

(C) $9x - 7y = 37$ (D) None of these

Sol. The given equations of the lines are:

$3x + 4y + 7 = 0$ and $12x + 5y - 4 = 0$

Making positive the constant term, we get

$\therefore 3x + 4y + 7 = 0, -12x - 5y + 4 = 0$

$\therefore$ Equation of angle of bisector in which origin does not lie is given by

$$\frac{3x+4y+7}{\sqrt{3^2+4^2}} = -\frac{-12x+5y+4}{\sqrt{(-12)^2+(-5)^2}} \text{ or, } \frac{3x+4y+7}{5} = -\left(\frac{-12x-y+4}{13}\right) \text{ or, } 7x - 9y = 37.$$

$\therefore$ The correct answer is (B).

FAMILIES OF LINES

The **joint equation** of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines if.

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad \textbf{OR}$$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Equation represents:

(i) a pair of lines passing through the origin if **$g = f = c = 0$ & $h^2 \geq ab$** (a, b and h are not all zero)

(ii) a pair of **straight lines** if $h^2 \geq ab$

(iii) a pair of **intersecting lines** if $h^2 > ab$

(iv) a pair of **parallel lines** / coincident if $h^2 = ab$.

Angle between pair of lines is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}, \quad (\theta \neq \pi/2)$$

Note:- Two line are perpendicular at right angles if

$a + b = 0$ or coeff. of x^2 + coeff of $y^2 = 0$

Relation between slopes:

If m_1 and m_2 be the slopes of the two lines given then

$$m_1 + m_2 = \frac{-2h}{b}, \; m_1m_2 = \frac{a}{b}$$

Pair of Bisectors:

The joint equation of the pair of bisectors of the angles between the lines is given by $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a-b} = \frac{xy}{h}, \quad h > ab.$$

Point of Intersection of the lines

The general equation of second-degree $ax^2 + 2hxy + by^2 + 2gh + 2fy + c = 0$ is generally denoted by $S = 0$.

$\therefore$ The point of intersection is $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$.

Examples:

Ex. Find the point of intersection of the lines represented by $2x^2 + 7xy + 3y^2 + 8x + 14y + 8 = 0$.

Sol. $2x^2 + 7xy + 3y^2 + 8x + 14y + 8 = 0$ … (1)

$\Rightarrow$ $2x^2 + (7y + 8)x + (3y^2 + 14y + 8) = 0$

$\Rightarrow$ $x = \dfrac{-(7y+8) \pm \sqrt{(7y+8)^2 - 8(3y^2+14y+8)}}{4}$

$\Rightarrow$ $4x + 7y + 8 = \pm\sqrt{49y^2 + 64 + 112y - 24y^2 - 112y - 64} = \sqrt{25y^2} = \pm 5y$

$\therefore$ $4x + 7y + 8 = 5y$ or $4x + 7y + 8 = -5y$

$\therefore$ The lines given by (1) are

$4x + 2y + 8 = 0$ … (2)

And $4x + 12y + 8 = 0$ … (3)

(2) – (3) $\Rightarrow$ $-10y = 0$ $\Rightarrow y = 0$

$\therefore$ (2) $\Rightarrow$ $4x + 2(0) + 8 = 0$ $\Rightarrow x = -2$

$\Rightarrow$ $\therefore$ The point of intersection is (– 2, 0).

Ex. If the pair of straight lines is given by $x^2 + 3xy + 2y^2 = 0$, then

(i) The given equation represents two real and distinct lines passing through origin.

(ii) The lines are $x + 2y = 0$, $x + y = 0$

(iii) The angles between the lines is $\tan^{-1}(1/3)$

Select the correct answer using the codes given below:

(A) (i) and (ii) (B) (i) and (iii) (C) (ii) and (iii) (D) (i), (ii) and (iii)

Sol. The given equation is $x^2 + 3xy + 2y^2 = 0$ $(x + 2y)(x + y) = 0$.

$\therefore$The equation represents two real, distinct lines whose equations are $x + 2y = 0$ and $x + y = 0$ each passing through origin.

If q be the angles between the lines, then

$$\theta = \tan^{-1} \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - 2}}{1+2} = \tan^{-1}\left(\frac{1}{3}\right).$$

$\therefore$All the statements i.e. (i), (ii) and (iii) are correct.

Therefore the correct answer of this question is (D).

Ex. The equation of bisector of angle of angle bisectors of the lines $ax^2 + 2hxy + by^2 = 0$ is:

(A) $h(x^2 - y^2) = (a - b)xy$ (B) $2x^2 + 2hxy + by^2 = 0$

(C) $(a - b)(x^2 - y^2) + 4hxy = 0$ (D) none of these

Sol. The given equation is $ax^2 + 2hxy + by^2 = 0$ … (1)

$\therefore$The equation of lines bisecting the angle between the given lines is

$\dfrac{x^2 - y^2}{a-b} = \dfrac{xy}{h}$ or $hx^2 - (a - b)xy - hy^2 = 0$ … (2)

$\therefore$The equation of lines bisecting the angle between the lines in (2) is:

$\dfrac{x^2 - y^2}{h-(-h)} = \dfrac{xy}{-\left(\frac{a-b}{2}\right)}$ or $(a - b)(x^2 - y^2) + 4hxy = 0$

Which is the required equation of bisectors of angle of bisectors of angle of equation (1)

$\therefore$The correct answer is (C).

Ex. Find the straight lines represented by the homogeneous equations :

(i) $33x^2 + 71xy - 14y^2 = 0$

(ii) $x^2 + 2xy\ \sec\theta + y^2 = 0.$

Sol. (i) The given equation is

$33x^2 + 71\,x\,y - 14y^2 = 0.$

$\Rightarrow 14\,y^2 - (71x)\,y - 33x^2 = 0$

$\Rightarrow\ y = \dfrac{71x \pm \sqrt{(71x)^2 + 4.14.33x^2}}{28}$

$= \dfrac{(71 \pm 83)x}{28} = \dfrac{154x}{28}, \dfrac{-12x}{28}$

$= \dfrac{11}{2}x, -\dfrac{3}{7}x$

$\therefore$ The lines are $y = \dfrac{11}{2}$ x and $y = \dfrac{-3}{7}$ x.

(ii) The given equation is

$x^2 + 2xy\sec\theta + y^2 = 0 \quad \Rightarrow y^2 + (2x\sec\theta)\,y + x^2 = 0$

$y = \dfrac{-2x\sec\theta \pm \sqrt{4x^2\sec^2\theta - 4}}{2} = -(\sec\theta \pm \tan\theta)\,x$

$\therefore\ y = -(\sec\theta - \tan\theta)\,x, -(\sec\theta + \tan\theta)\,x$

$\therefore$ The lines are $y = -(\sec\theta - \tan\theta)\,x$ and $y = -(\sec\theta + \tan\theta)\,x$.

Ex. The angle between the lines represented by the equation

$2x^2\cos\alpha - \sin^2\alpha\,x\,y - y^2\,\text{cosec}\,\alpha\,\sin 2\alpha + 3 = 0$ is:

(A) $\pi/4$ (B) $\pi/3$ (C) $\pi/2$ (D) 0

Sol. The given equation is $2x^2\cos\alpha - xy\sin^2\alpha - y^2\,\text{cosec}\,\alpha\,\sin 2\alpha + 3 = 0$

$\Rightarrow 2x^2\cos\alpha - xy\sin^2\alpha - 2y^2\cos\alpha + 3 = 0$

$\Rightarrow$ Here $a = 2\cos\alpha,\ h = -\dfrac{\sin^2\alpha}{2},\ b = -2\cos\alpha,\ c = 3$

Because $a + b = 2\cos\alpha - 2\cos\alpha = 0.$

Therefore the lines are perpendicular i.e., $\theta = \pi/2$.

$\therefore$ The correct answer is (C).

Ex. If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is $\tan^{-1}$ (m), then m is equal to:

(A) 1/5 (B) – 1 (C) – 2/3 (D) none of these

Sol. The angle between the lines $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is given by

$\tan\theta = \pm\dfrac{2\sqrt{\frac{25}{4} - 6}}{2 + 3} = \pm\dfrac{1}{5}\ \therefore \theta = \tan^{-1}\left(\pm\dfrac{1}{5}\right)$. **Answer: (A)**

1. Distance formula	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Section formula	$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$ Internally $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$ externally
3. In center of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) and a, b , c as sides	$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$
4. Centroid of a triangle with vertices (x_1, y_1), (x_2 , y_2) and (x_3, y_3)	$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$,
5. Area of triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3)	$\frac{1}{2}$ mod of $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$
6. One point form	$y - y_1 = m (x - x_1)$
7. Slope intercept form	$y = m x + c$
8. Intercept form	$\frac{x}{a} + \frac{y}{b} = 1.$
9. Two point form	$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1), x_1 \neq x_2$
10. Distance form	$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$
11. Normal form	$x \cos \alpha + y \sin \alpha = p$
12. Angle between two lines	$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2}$
13. Distance between || lines ax + by + c = 0 and ax + by +d =0	$\frac{\lvert c - d \rvert}{\sqrt{a^2 + b^2}}$
14. Any line parallel to a x + by + c = 0 is	ax + by + k = 0
15. Any line $\perp$ to a x + by + c = 0 is	bx - ay + k = 0
16. Condition for the Concurrency of three straight lines	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$
17. Bisector of Angles (note: if $a_1 a_2 + b_1 b_2 < 0$ + ve sign for acute angle - ve for obtuse if $a_1 a_2 + b_1 b_2 > 0$ +ve sign for obtuse angle, -ve for acute angle.) In which origin lies take + ve sign	$\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right)$